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GENERALIZED J-INTEGRAL AND ITS APPLICATION

— An interpretation of Hadamard's variational formula —

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§1. INTRODUCTION. In Eshelby[2], J-integral (Energy Momentum Tensor) is applied for a study of continuum theory of lattice defects. Two dimensional version of J-integral is connected in Rice[16] with the energy release rate of straight crack extension. In absence of defects, Günther[8] and Knowles & Sternberg[10] add two new surface integrals to J-integral, called L,M-integral, whose physical meanings are conservation laws obtained by Noether's theorem[12] on variational principle. Furthermore, the three conservation laws mentioned above are complete within the context of linear isotropic, homogeneous elastostatics, in the sense that they are obtained by Noether's scheme (see [10]). These surface integrals are written as follows:

Let Ω be an open set in $\mathbb{R}^3$ occupied by the elastic body in its non-deformed state. We say that a triple $[u,\varepsilon,\sigma]$ is an elastic state if $[u,\varepsilon,\sigma]$ is, in this order, the displacement vector, the strain tensor and the stress tensor all of which are taken to be defined on Ω. The material considered here is homogeneous (nonlinear) elasticity, i.e., there is a strain energy density W and the stress-strain relation is given by the formula $\sigma_{ij} = \partial W/\partial \varepsilon_{ij}$. Let S be a closed surface contained in Ω. We assume that a interior force is zero in the domain enclosed by S. If $[u,\varepsilon,\sigma]$ is so smooth that the divergence theorem may be applicable, then we have

$$(1.1) \qquad J_k \equiv \int_S \{W\nu_k - T \cdot D_k u\}\, dS = 0 \quad \text{for} \quad k = 1,2,3,$$

which is also derived as a consequence of the invariance of the potential energy under a coordinate translation. Here ν_k are the components of the outward unit normal to S, T the traction vector on S, i.e., $T_i = \sigma_{ij}\nu_j$ and dS the surface element of S. Moreover, if the material considered is isotropic, then the potential energy is also invariant under the rotation of coordinates. Hence we have

$$(1.2) \qquad L_\alpha \equiv \int_S \varepsilon_{\alpha\ell k}\{Wx_k\nu_\ell - T_\ell u_k - T\cdot(D_\ell u)x_k\}\, dS = 0$$

for $\alpha = 1,2,3$, where $\varepsilon_{\alpha\ell k}$ are the components of the antisymmetric third order tensor such that $\varepsilon_{123} = +1$. The third conservation law for linear elastostatics is derived as a result of infinitesimal invariance of potential energy under a coordinate scale change, which is written as

$$(1.3) \qquad M \equiv \int_S \{Wx_i\nu_i - T\cdot(D_i u)x_i - (T\cdot u)/2\}\, dS = 0.$$

If there are defects (singularity) within S, then these conservation laws will not hold (see e.g. Budiansky & Rice[1]). These results indecate that, in J, L and M-integrals, there exist three physical meanings

(1) These quantities represent a class of consevation laws obtained by Noether's theorem.

(2) These quantities are forces acting on singularity within S.

(3) In two dimensional case, J_1-integral equals the energy release rate of straight crack extension.

But forces acting on the singularity are not only these quantities. Indeed, for three dimensional crack problem, there is a crack extension whose energy release rate cannot be expressed by these surface integrals (see Ohtsuka[13]). Generalized J-integral (GJ-integral) is proposed as a generalization of these surface integrals in order to express the energy release rate of smooth crack extension (see [13], Definition 3.3).

We may consider that Hadamard's variational formula (see Theorem 3.1) express a behavior of solutions at the boundaries. But, in the total space, the boundary is the set of singularities of solutions. In section three, we reconsider Hadamard's variational formula from this viewpoint and apply GJ-integral to obtain it. As shown in section four, such a point of view is important when we study the mixed boundary value problems. We now generalize GJ-integral to be applicable for various elliptic boundary value problems in the next.

§2. GJ-INTEGRAL AND ITS PHYSICAL MEANING. Let Ω be a bounded domain in $\mathbb{R}^n$ and W be a given function in $C^1(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$. We consider a functional of potential type $I(v;\Omega)$ defined on a Hilbert space $V(\Omega)$ (contained in $H^1(\Omega)$)

$$I(v;\Omega) = \int_\Omega \{W(x,v,\nabla v) - f\cdot v\}\, dx$$

for a given function f in $L^2(\Omega)$. Here we assume that $V(\Omega)$ is dense in $L^2(\Omega)$.

The problem we now consider is the following

P: Find $u \in V(\Omega)$ such that $I(u;\Omega) \leqq I(v;\Omega)$ for all $v \in V(\Omega)$.

Hereafter we assume that the problem P is uniquely solvable, and suppose that there exists a limit

$$DW(v,h) = \lim_{\varepsilon\to 0} \varepsilon^{-1}[W(v+\varepsilon h) - W(v)]$$

for all $v,h \in V(\Omega)$. Then the solution u satisfies that

$$\int_\Omega DW(u,h)\,dx = \int_\Omega f\cdot h\,dx \quad \text{for all} \quad h \in V(\Omega).$$

If u and h are suitably smooth, then the following Green's formula hold

$$\int_\Omega DW(u,h)\,dx = -\int_\Omega Au\cdot h\,dx + \int_{\partial\Omega} N(u)\cdot h\,dS \tag{2.1}$$

where $u \to Au$ is a partial differential (nonlinear) operator on of order 2, and $u \to N(u)$ is a partial differential (nonlinear) operator on $\partial\Omega$ of order 1.

Let A be a domain in $\mathbb{R}^n$ and let $X(A)$ be a set of all suitably smooth vector fields defined on $\bar{A}$. We call a domain A "regular relative to Ω" if A is a bounded closed domain in $\mathbb{R}^n$ and the divergence theorem holds on $A \cap \Omega$, for all suitably smooth functions and all elements in $X(A)$. Let us define GJ-integral $J_A(u;\mu)$ as a functional of all domains A in $\mathbb{R}^n$ regular relative to Ω, all solution u of P and μ in $X(A)$.

<u>Definition 2.1.</u> If the following two quantities

$$P_A(u;\mu) = \int_S \{W(u)(\mu\cdot\nu) - N(u)X_\mu(u)\}\,dS,$$

$$R_A(u;\mu) = -\int_{A'} \{X_\mu(W(u)) - DW(u,X_\mu(u)) + W(u)\operatorname{div}\mu + f\cdot X_\mu(u)\}\,dx$$

are finite, then GJ-integral $J_A(u;\mu)$ is defined by

$$J_A(u;\mu) = P_A(u;\mu) + R_A(u;\mu)$$

where S is the boundary of A'; $A' = A \cap \Omega$, ν the outward unit normal to S, $(\mu\cdot\nu)$ the inner product of ν and μ; $\mu \in \mathcal{X}(A)$, and $X_\mu = \mu\cdot\nabla$.

<u>Example 2.1.</u> If $2W(u) = |\nabla u|^2 + pu^2$ (Laplace equation), then $N(u) = \partial u/\partial\nu$; $X_\mu(W)-DW(u,X_\mu u) = -D_j u(D_j\mu^h)D_h u$.

<u>Example 2.2.</u> If $2W(u) = a_{ij}D_i uD_j u + au^2$, then $N(u) = a_{ij}\nu_j D_i u$; $X_\mu(W)-DW(u,X_\mu u) = X_\mu(a_{ij}/2)D_i uD_j u - a_{ij}D_i u(D_j\mu^h)D_h u$.

<u>Example 2.3.</u> If $2W(u) = \sigma_{ij}\varepsilon_{ij}$ (linear elasticity), where $\varepsilon_{ij} = (D_j u_i+D_i u_j)/2$, $\sigma_{ij} = c_{ijk\ell}\varepsilon_{k\ell}$, then $N(u) = \sigma_{ij}\nu_j$; $X_\mu(W)-DW(u,X_\mu u) = X_\mu(c_{ijk\ell}/2)\varepsilon_{k\ell}\varepsilon_{ij}-\sigma_{ij}(D_j\mu^h)(D_h u_i)$.

<u>Example 2.4.</u> If $W(u)$ is given as a strain energy density, and if a genergal nonlinear Hooke's law $\sigma_{ij} = \partial W/\partial\varepsilon_{ij}$ holds, then $N(u) = \sigma_{ij}\nu_j$; $X_\mu(W)-DW(u,X_\mu u) = -\sigma_{ij}(D_j\mu^h)(D_h u)$.

<u>Remark 2.1.</u> If $DW(u,h)$ is a bilinear form, then $u \to R_A(u)$ is a bounded functional on $V(\Omega)$ (see e.g. Examples 2.1-3). In Example 2.4, $u \to R_A(u)$ is also bounded on $\{H^1(\Omega)\}^3$ in a special case of the nonlinear Hooke's law (see Necas & Hlavacek[11], Chapter 8).

<u>Therorem 2.1.</u> If a solution u of P is so smooth that the divergence theorem may be applicable on $A \cap \Omega$ and $\mu \in \mathcal{X}(A)$, then $J_A(u,\mu) = 0$.

<u>Proof.</u> By the divergence theorem we have

$$\int_{A'} X_\mu(W(u))\, dx = \int_S W(u)(\mu\cdot\nu)\, dS - \int_{A'} W(u)\mathrm{div}\mu\, dx.$$

Moreover by Green's formula it follows that

$$\int_{A'} DW(u,X_\mu u)\,dx = \int_{A'} f\cdot X_\mu(u)\,dx + \int_S N(u)X_\mu(u)\,dS.$$

Thus we have that $J_A(u;\mu) = 0$.

Theorem 2.2. Let u be a solution of the problem P whose W is given in Example 2.4. Assume that $f = 0$ on A, and $[u,\varepsilon,\sigma]$ is so smooth that the divergence theorem may be applicable on $A \cap \Omega$ and $X(A)$, then the surface integral (1.1) is written as

$$J_k = J_A(u;e_k) \quad \text{for} \quad k = 1,2,3$$

with the unit base vector e_k in the x_k-direction.

In addition, if $[u,\varepsilon,\sigma]$ is isotropic, then

$$L = J_A(u;q_\alpha) \quad \text{for} \quad \alpha = 1,2,3$$

$$\text{with} \quad q_\alpha(x) = (\varepsilon_{\alpha 1k}x_k, \varepsilon_{\alpha 2k}x_k, \varepsilon_{\alpha 3k}x_k).$$

Finally, if $[u,\varepsilon,\sigma]$ is linear (not necessarily isotropic), then

$$M = J_A(u;x) \quad \text{with} \quad x = (x_1,x_2,x_3).$$

Proof. For the proof we may refer to [13], Theorem 3.5.

Physical meaning of GJ-integral Let G be a domain with smooth boundary. Let us assume that $\Omega = G - \Sigma$ in which Σ means defects. The initial defects Σ move smoothly depending on time t, denoted by $\Sigma(t)$. We denote by $\Omega(t)$ the set $G - \Sigma(t)$. Here we assume that there eixts a family of C^∞-diffeomorphisms $\phi_t\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\phi_t(\Omega) = \Omega(t)$. Let us set

$$\mu(x) = \frac{d}{dt}\phi_t(x)\Big|_{t=0}.$$

Then by a formal calculation we have

(2.2) $$-J_A(u;\mu) = \lim_{t\to 0} t^{-1}[I(\chi_A u;\Omega(t)) - I(\chi_A u;\Omega)]$$

where χ_A is the characteristic function of A, i.e., $\chi_A = 1$ on A and $\chi_A = 0$ on A^c.

This indicates that $J_A(u;\mu)$ equals the rate of energy change by force acting across S and acting inside the domain enclosed by S. In the case when $\{\Sigma(t)\}$ is a family of smooth cracks, the precise proof of (2.2) is given in [13].

§3. A INTERPRETATION OF HADAMARD'S VARIATIONAL FORMULA

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary γ. Let $\rho(x)$ be a smooth function defined on γ and ν_x the outward unit normal to γ. For any sufficiently small $t \geqq 0$, let $\Omega(t)$ be the bounded domain whose boundary $\gamma(t)$ is defined by $\gamma(t) = \{x+t\rho(x)\nu_x; x \in \gamma\}$. Consider Green's function $G_t(x,y)$ of the Laplacian with the Dirichlet condition on $\gamma(t)$, i.e.,

$$\Delta_x G_t(x,y) = -\delta(x-y) \quad \text{for} \quad x,y \in \Omega(t),$$

$$G_t(x,y)\big|_{x \in \gamma(t)} = 0 \quad \text{for} \quad y \in \Omega(t),$$

where δ is the Dirac function and we set $G = G_0$.

<u>Theorem 3.1.</u> (Hadamard's variational formula[9]) For fixed $x,y \in \Omega$, there exists $\delta G(x,y) = (\partial G_t(x,y)/\partial t)_{t=0}$. Furthermore we have

(3.1) $$\delta G(x,y) = \int_\gamma \frac{\partial G(x,z)}{\partial \nu_z} \frac{\partial G(y,z)}{\partial \nu_z} \rho(z)\, dS_z.$$

In total space $\mathbb{R}^n$, the boundary γ of Ω is the set of singularity of the solution of the Dirichlet problem for Laplace's equation $-\Delta u = f$ in Ω. The formula (3.1) is regarded to represents the behavior of this singularity. We now derive (3.1) by GJ-integral. Let us consider a family of boundary value problems.

P_t: For a given f in $L^2(\Omega)$, find u_t in $H_0^1(\Omega)$ which minimize the potential energy functional

$$I(v;\Omega(t)) = \int_{\Omega(t)} \{\frac{1}{2}|\nabla v|^2 - f\cdot v\}\, dx$$

on $H_0^1(\Omega)$.

The problem P_t is uniquely solvable, whence there exits a family of Green's operators

$$G_t\colon f \to u_t = G_t f.$$

We now consider a family of bounded operators in $L^2(\Omega)$ defined by

$$\hat{G}_t f = r_\Omega\{\text{zero extension of } G_t[r_{\Omega(t)}\tilde{f}] \text{ outside } \Omega(t)\}$$

where $\tilde{f}$ denotes the zero extension of f outside Ω and $r_{\Omega(t)}$ the restriction operators. Since the adjoint operator of $r_{\Omega(t)}$ is the zero extension outside $\Omega(t)$ for each t, we have for $g \in L^2(\Omega)$

$$(\hat{G}_t f,g)_\Omega = (G_t[r_{\Omega(t)}\tilde{f}],r_{\Omega(t)}\tilde{g})_{\Omega(t)}$$

where

$$(f,g)_{\Omega(t)} = \int_{\Omega(t)} f\cdot g\, dx.$$

So that G_t is self-adjoint in $L^2(\Omega)$, since G_t is self-adjoint in $L^2(\Omega(t))$ for each t, i.e.,

$$(\hat{G}_t f,g)_\Omega = ([r_{\Omega(t)}\tilde{f}],G_t r_{\Omega(t)}\tilde{g})_{\Omega(t)} = (f,G_t g)_\Omega.$$

Hence the operator $t^{-1}[G_t - G]$ is a bounded self-adjoint operator on $L^2(\Omega)$ for each $t > 0$. If

(3.2) $\quad \lim_{t\to 0} (t^{-1}[\hat{G}_t - G]f,f)_\Omega$ exists for all f in $L^2(\Omega)$,

then from Banach-Steinhaus theorem we have a bounded linear operator δG on $L^2(\Omega)$ such that

$$t^{-1}[\hat{G}_t - G] \to \delta G \quad \text{in} \quad \mathcal{B}(L^2(\Omega) \to L^2(\Omega)) \quad \text{weakly as} \quad t \to 0,$$

where $\mathcal{B}(L^2(\Omega) \to L^2(\Omega))$ is the space of all bounded linear operators on $L^2(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, the condition (3.2) is equivalent to the following

(3.3) $\quad \lim_{t\to 0} \int_\Omega t^{-1}(u_t - u)\cdot f\, dx$ exists for all $f \in C_0^\infty(\Omega)$,

where u_t is the solution of P_t for the given f in $C_0^\infty(\Omega)$. On the other hand the following hold

$$\int_\Omega t^{-1}(u_t-u)\cdot f\, dx = t^{-1}\{\int_{\Omega(t)} |\nabla u_t|^2\, dx - \int_\Omega |\nabla u|^2\, dx\}$$
$$= t^{-1}\{2[I(u_t;\Omega(t)) - I(u;\Omega)] + 2\int_\Omega t^{-1}(u_t-u)\cdot f\, dx\}$$

whence we can deduce that

$$\lim_{t\to 0} t^{-1}\int_\Omega (u_t-u)\cdot f\, dx = 2\{\lim_{t\to 0} t^{-1}[I(u;\Omega) - I(u_t;\Omega(t))]\}.$$

Thus the condition (3.2) is equivalent to the existence of the variation of potential energy with respect to the boundary perturbation.

Consider C^∞-differomorphisms ϕ_t from $\mathbb{R}^n$ onto $\mathbb{R}^n$ defined by

$$\phi_t(x) = x + t\beta(x)\rho(x)\nu_x$$

where $\beta \in C_0^\infty(R^n)$ satisfies $\beta \geqq 0$ and $\beta = 1$ near $\gamma(t)$ for all t. Then $\mu(x) = \beta(x)\rho(x)\nu_x$.

Let A be an open set in $\mathbb{R}^n$ such that $A \supset \Omega(t)$ for all t. Since

$$I(u_t;\Omega(t)) = \int_A \{\tfrac{1}{2}|\nabla u_t|^2 - f\cdot u_t\}\,dx$$

it follows from the physical meaning of GJ-integral (section two) that

$$\lim_{t\to 0}\int_\Omega t^{-1}(u_t-u)\cdot f\,dx = 2J_A(u;\mu).$$

Since $u = 0$ on $A-\Omega$, we obtain that $J_A(u;\mu) = R_\Omega(u;\mu)$. But by the regularity theorem it follows that $J_\Omega(u;\mu) = 0$. Thus we have

$$\begin{aligned}(\delta Gf,f)_\Omega &= -2P_\Omega(u;\mu)\\ &= 2\int_\gamma \{(\partial u/\partial\nu)^2 - \tfrac{1}{2}|\nabla u|^2\}\rho\,dS.\end{aligned}$$

Observing that $u = 0$ on γ, we have $|\nabla u|^2 = (\partial u/\partial\nu)^2$, whence

$$(\delta Gf,f)_\Omega = \int_\gamma (\partial u/\partial\nu)^2\rho\,dS,$$

from which it follows that

$$(\delta Gf,g)_\Omega = \int_\gamma (\partial u/\partial\nu)(\partial v/\partial\nu)\rho\,dS$$

where $u = Gf$ and $v = Gg$.

Therefore by the kernel representation of Green's operator G, the Lebesgue convergence theorem and Fubini's theorem, we can derive the following

$$(\delta Gf,g)_\Omega = \int_\Omega g(x)\,dx\int_\Omega f(y)\,dy\int_\gamma \frac{\partial G(x,z)}{\partial\nu_z}\,\frac{\partial G(y,z)}{\partial\nu_z}\rho(z)\,dS_z.$$

Since we can take arbitrary functions in $C_0^\infty(\Omega)$ for f,g, Theorem 3.1 follows.

Next we consider Neumann's function G^N associated with the operator $-\Delta + p$ $(p \geqq c > 0)$. In this case Hadamard's variational formula is the following

Theorem 3.2. (see Garabedian & Schiffer[5])

$$\delta G^N(x,y) = -\int_\gamma \{\nabla G^N(x,z)\nabla G^N(y,z) + p(z)G^N(x,z)G^N(y,z)\}\rho(z)\, dS_z. \tag{3.4}$$

In the case of the Dirichlet problem, GJ-integral acts on the singularity of zero extension. But, for the Neumann problem, we must consider another extension.

We now consider the following singular perturbation of the Neumann problem:

P^ε: Find u^ε in $H^1(R)$ such that for a given f in $C_0^\infty(\Omega)$

$$\int_\Omega \{\nabla u^\varepsilon \nabla\varphi + pu^\varepsilon\varphi\}dx + \int_{\Omega_0} \varepsilon\{\nabla u^\varepsilon\nabla\varphi + pu^\varepsilon\varphi\}\, dx = \int_\Omega f\cdot\varphi\, dx$$

hold for all $\varphi \in H^1(R)$, where R is a domain in R^n with smooth boundary satisfying that $\bar{\Omega} \subset R$ and $\Omega_0 = R - \bar{\Omega}$.

A solution of the problem P^ε is the solution of the following transmission problem

$$-\Delta u^\varepsilon + pu^\varepsilon = f \quad \text{in } \Omega; \quad -\Delta u^\varepsilon + pu^\varepsilon = 0 \quad \text{in } \Omega_0,$$

$$(u^\varepsilon)^+ = (u^\varepsilon)^-, \ (\partial u^\varepsilon/\partial\nu)^+ = \varepsilon(\partial u^\varepsilon/\partial\nu)^- \quad \text{on } \partial\Omega,$$

$$\partial u^\varepsilon/\partial\nu = 0 \quad \text{on } \partial\Omega_0 - \partial\Omega,$$

where ν is the outward unit normal to $\partial\Omega$ (or $\partial\Omega_0 - \partial\Omega$) and for $h \in H^1(R)$,

$$h^+ (\text{resp. } h^-) = \text{trace of } h|_\Omega \ (\text{resp. } h|_{R-\bar{\Omega}}) \ \text{on } \Omega.$$

Then we have

Theorem 3.3. Let u^0 be a solution of the Neumann problem, i.e.,

$$-\Delta u^0 + pu^0 = f \quad \text{in} \quad \Omega; \ \partial u^0/\partial\nu = 0 \quad \text{on} \quad \partial\Omega.$$

Then the solution u^ε of the problem P^ε converges to u^0 in the following manner:

$$u^\varepsilon \to u^0 \quad \text{in} \quad H^2(\Omega) \quad \text{as} \quad \varepsilon \to 0; \ |u^\varepsilon|_{2,\Omega_0} \leqq \text{const.}, \ \forall\varepsilon \geqq 0.$$

Furthermore, there exists a function v in $H^2(\Omega_0)$ and a subsequence $\{\varepsilon_j\}$ of $\{\varepsilon\}$ such that

$$u^{\varepsilon_j} \to v \quad \text{in} \quad H^\alpha(\Omega_0) \quad \text{as} \quad j \to \infty \quad \text{for all} \quad \alpha < 2,$$

and $(u^0)^+ = v^-$ on $\partial\Omega$.

However, no proof of this statement will be given here (see [14]).

If, for each $\varepsilon > 0$, we perturb the interface $\gamma = \partial\Omega$ by a family of surfaces $\gamma(t)$, then the behavior of solutions determined by this perturbation is described as GJ-integral which acts on the singularities at the interface. Letting $\varepsilon \to 0$, we get by Theorem 3.3 the formula

$$\lim_{t\to 0}\int_\Omega t^{-1}(u_t^0 - u^0)\cdot f\,dx = \int_\gamma \{2(\partial u^0/\partial\nu)^2 - |\nabla u^0|^2 - p(u^0)^2\}\rho\,dS.$$

Since $\partial u^0/\partial\nu = 0$ on γ, Theorem 3.2 follows from a similar argument as in the case of the Dirichlet problem.

In the next section we shall show Hadamard's variational formula without making use of extension operators.

§4. HADAMARD'S VARIATIONAL FORMULA.

In preceding we studied the case when a weak solution of the problem is also a strong solution, so that the corresponding Green's functions are sufficiently smooth. In such a case Fujiwara & Ozawa[3] generalize the works of Hadamard[9], Garabedian & Schiffer[5] and Garabedian[4] to the case of Green's function of some normal elliptic boundary value problems by means of the Whitney extension theorem. Another approach is made in Peetre[15]. If a weak solution cannot be a strong solution, then GJ-integral will be useful to obtain Hadamard's variational formula. We here consider the following problems.

(1) Does Hadamard's variational formula hold for a perturbation of domains with local Lipschitz property ?

(2) Does Hadamard's variational formula hold for the mixed boundary value problems ?

Theorem 4.1. Let Ω be a doamain with local Lipschitz property. Assume that a deformation $\Omega(t)$ of Ω is written as $\Omega(t) = \phi_t(\Omega)$ by a family $\{\phi_t\}$ of C^∞-diffeomorphisms from $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that ϕ_t: $[0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is of class of C^∞. Suppose that the mapping $v(x) \to \phi_t^* v(x) = v(\phi_t(x))$ is a continuous bijection from $V(\Omega(t))$ onto $V(\Omega)$, and suppose that $\int_{\Omega(t)} DW(v,h)\, dx$ is a strictly coercive bilinear form. Then

$$\lim_{t\to 0} t^{-1}[I(u;\Omega) - I(u(t);\Omega(t))] = R_\Omega(u;\mu).$$

The proof is similar to that of [13], Theorem 4.5.

Remark 4.1. In the mixed boundary value problem, it should be noticed that a perturbation applies not only to the domain, but also to the interface between the Dirichlet part and the Neumann part. Here the Dirichlet part $\gamma_D(t)$ of $\gamma(t)$ has positive measure and written as $\gamma_D(t) = \phi_t(\gamma_D)$; $\gamma_D = \gamma_D(0)$.

<u>Remark 4.2.</u> Notice that $u \to R_A(u)$ is bounded in $H^1(\Omega)$.

<u>Remark 4.3.</u> If a domain Ω with non-smooth boundary is perturbed by smooth domanis $\Omega(t)$, then the above theorem does not hold.

<u>Theorem 4.2.</u> Let $u(t)$ be a solution of the problem P_t under the hypothesis in Theorem 4.1. If u belongs to $H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$, then

$$R_\Omega(u;\mu) = -J_\Omega(u;\mu).$$

If Ω is convex, then a solution u of the Dirichlet (resp. Neumann) problem for Laplacian belongs to $H^2(\Omega)$ (see e.g. Grisvard[7]). Hence

<u>Corollary 4.2.1.</u> If Ω is convex, then (3.1) and (3.4) are valid under the hypothesis in Theorem 4.1.

If Ω is a polygonal domain in $\mathbb{R}^2$, then a solution u of the Dirichlet (resp. Neumann) problem for Laplacian belongs to $H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ (see e.g. Grisvard[6]). Hence

<u>Corollary 4.2.2.</u> If Ω is polygonal domain, then (3.1) and (3.4) are valid for a perturbation satisfying the hypothesis in Theorem 4.1.

Finally we study the mixed boundary value problem.

Let $\Omega = \mathbb{R}^2_+$. Let $\gamma_D = \{x \in \mathbb{R}^2;\ x_2 = 0,\ x_1 > 0\}$ and let $\gamma_N = \partial\Omega - \overline{\gamma_D}$. Then the weak solution u of the mixed boundary value problem for Laplacian

$$-\Delta u = f \quad \text{in} \quad \Omega \quad \text{for a given} \quad f \quad \text{in} \quad L^2(\Omega),$$

$$u = 0 \quad \text{on} \quad \gamma_D; \quad \partial u/\partial x_2 = 0 \quad \text{on} \quad \gamma_N \quad \text{and} \quad u = 0 \quad \text{for} \quad |x| > 1,$$

is decomposed as follows

$$u(x) = c\zeta(x) r^{1/2}\sin(\theta/2) + w(x); \; r = |x|, \; \theta = \tan^{-1}(x_2/x_1)$$

where c is a constant determined by f, w belongs to $H^2(\Omega)$ and ζ $C_0^\infty(R^2)$ such that $\mathrm{supp}\zeta \subset \{x; \; |x| < 1\}$, $\zeta \geqq 0$ and $\zeta = 1$ near the origin. Here we notice that $c\zeta(x)r^{1/2}\sin(\theta/2)$ does not belong to $H^{3/2}(\Omega)$ but belongs to $H^{3/2-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. Let us put $\mu(x) = \alpha e_1 + \beta e_2$, where e_i is the base vector in the x_i-direction, and α, β are constants.

We shall calculate the typical case when only the interface is perturbed (see Figure 4.1). Since the solution u is regular except at the origin, we then have

$$R_\Omega(u;\mu) = R_\Omega(u;\mu) - J_{\Omega-B(r)}(u;\mu)$$

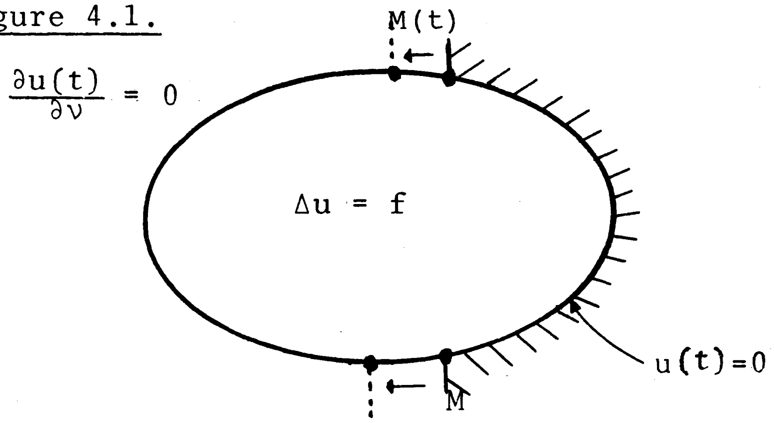
for all $r > 0$, where $B(r) = \{x; \; |x| < r\}$. Hence

$$R_\Omega(u;\mu) = \lim_{r\to 0} \int_0^\pi \{\frac{1}{2}|\nabla u|^2\cos\theta - \frac{\partial u}{\partial r}[\frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial\theta}\cos\theta]\} r\,d\theta$$

$$= \frac{1}{8}\alpha c^2\pi.$$

This is an analoge of Hadamard's variational formula under the perturbation of the interfaces (see Figure 4.1).

Figure 4.1.



A perturbation $\{M(t)\}$ of the interface M

$$M(t) = \bar{\gamma}_D(t) \cap \bar{\gamma}_N(t)$$

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